Noncommutative differential geometry based on ε-derivations

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0804.3061 [hep-th], 0811.3567 [math-ph], 0811.3850 [math-ph]
Journ. N. C. Geom., SIGMA, CMP to appear

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Integrable Systems and Quantum Symmetries, Prague, 18-20 June 2009

Content

Noncommutative geometry based on ϵ-derivations

Commutation factor

- Let $\mathbb{K}$ a field and Γ an abelian group.

Definition 1 (Commutation factor)

A commutation factor on Γ over $\mathbb{K}$ is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ satisfying, $\forall i, j, k \in \Gamma$
$\varepsilon(i,j)\varepsilon(j,i) = 1_{\mathbb{K}}$, $\varepsilon(i, j+k) = \varepsilon(i,j)\varepsilon(i,k)$, $\varepsilon(i+j,k) = \varepsilon(i,k)\varepsilon(j,k)$

- This definition implies that $\varepsilon(i,0) = \varepsilon(0,i) = 1_{\mathbb{K}}$, $\varepsilon(i,i) \in \{1_{\mathbb{K}}, -1_{\mathbb{K}}\}$ and $\varepsilon(j,i) = \varepsilon(i,-j) = \varepsilon(i,j)^{-1}$
- Example : $\Gamma = \mathbb{Z}$, $\varepsilon(p,q) = 1$, $\varepsilon(p,q) = (-1)^{pq}$

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Proposition 2

Let ε_1 and ε_2 be two commutation factors respectively on the abelian groups Γ_1 and Γ_2, over the same field $\mathbb{K}$. Then the relation

$$\varepsilon((i_1,i_2),(j_1,j_2)) = \varepsilon_1(i_1,j_1)\varepsilon_2(i_2,j_2)$$

$\forall i_1, j_1 \in \Gamma_1$ and $\forall i_2, j_2 \in \Gamma_2$, defines a commutation factor on the abelian group $\Gamma_1 \times \Gamma_2$

ϵ-algebras

- Recall (Scheunert 1979, ...) : An ε-Lie algebra is defined as a couple $(\mathfrak{g}^\bullet, [-,-]_\varepsilon)$ where $\mathfrak{g}^\bullet$ is a Γ-graded vector space, ε a commutation factor on Γ and $[-,-]_\varepsilon : \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \to \mathfrak{g}^\bullet$ a bilinear product satisfying $[a,b]_\varepsilon = -\varepsilon(|a|,|b|)[b,a]_\varepsilon$, $\quad [a,[b,c]_\varepsilon]_\varepsilon = [[a,b]_\varepsilon, c]_\varepsilon + \varepsilon(|a|,|b|)[b,[a,c]_\varepsilon]_\varepsilon$
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Definition 3 (ε-graded algebra)

Let $\mathbf{A}^\bullet$ be an associative unital Γ-graded $\mathbb{K}$-algebra and a commutation factor ε on Γ, then $(\mathbf{A}^\bullet, \varepsilon)$ will be called an ε-graded (associative) algebra.

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- $(\mathbf{A}^\bullet, [-,-]_\varepsilon)$ is an ε-Lie algebra denoted by $\mathbf{A}^\bullet_{\text{Lie},\varepsilon}$
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- The ε-center of $\mathbf{A}^\bullet_\varepsilon$ is the ε-graded commutative algebra $\mathcal{Z}^\bullet_\varepsilon(\mathbf{A}) = \{a \in \mathbf{A}^\bullet,\ \forall b \in \mathbf{A}^\bullet\ [a,b]_\varepsilon = 0\}$

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- A Γ-graded module $\mathbf{M}^\bullet$ on $\mathbf{A}^\bullet$ is a Γ-graded vector space and a module on $\mathbf{A}^\bullet$ such that $\mathbf{M}^i\mathbf{A}^j \subset \mathbf{M}^{i+j}$ (for right modules) $\forall i, j \in \Gamma$

ϵ-derivations

Definition 4 (ε-derivation)

A linear map $\mathfrak{X} : \mathbf{A}^\bullet \to \mathbf{A}^\bullet$ is an homogeneous ε-derivation if, $\forall a, b \in \mathbf{A}^\bullet$,
$\mathfrak{X}(a{\cdot}b) = \mathfrak{X}(a){\cdot}b + \varepsilon(|\mathfrak{X}|, |a|)\ a{\cdot}\mathfrak{X}(b)$ where a are homogeneous.
We note $\mathrm{Der}^\bullet_\varepsilon(\mathbf{A})$ the Γ-graded space of ε-derivation

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- The space $\mathrm{Der}_\varepsilon^\bullet(\mathbf{A})$ is an ε-Lie algebra and a $\mathcal{Z}_\varepsilon^\bullet(\mathbf{A})$-module
- An inner ε-derivation on $\mathbf{A}^\bullet$ is an ε-derivation $\mathfrak{X}$:
$$\exists a \in \mathbf{A}^\bullet,\ \mathfrak{X}(b) = \mathrm{ad}_a(b) = [a, b]_\varepsilon$$
We note $\mathrm{Int}_\varepsilon^\bullet(\mathbf{A}) = \{\mathrm{ad}_a,\ a \in \mathbf{A}^\bullet\}$ the space of inner ε-derivations on $\mathbf{A}^\bullet$
It is an ε-Lie ideal and a $\mathcal{Z}_\varepsilon^\bullet(\mathbf{A})$-module.

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It is an ε-Lie ideal and a $\mathcal{Z}^\bullet_\varepsilon(\mathbf{A})$-module.
- Two short exact sequences of ε-Lie algebras and $\mathcal{Z}^\bullet_\varepsilon(\mathbf{A})$-modules :

$$0 \longrightarrow \mathcal{Z}^\bullet_\varepsilon(\mathbf{A}) \longrightarrow \mathbf{A}^\bullet \xrightarrow{\mathrm{ad}} \mathrm{Int}^\bullet_\varepsilon(\mathbf{A}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Int}^\bullet_\varepsilon(\mathbf{A}) \longrightarrow \mathrm{Der}^\bullet_\varepsilon(\mathbf{A}) \longrightarrow \mathrm{Out}^\bullet_\varepsilon(\mathbf{A}) \longrightarrow 0$$

Differential calculus based on ϵ-derivations

- $\underline{\Omega}_\varepsilon^{n,k}(\mathbf{A},\mathbf{M})$ is the space of n-linear maps ω from $(\mathrm{Der}_\varepsilon^\bullet(\mathbf{A}))^n$ to $\mathbf{M}^\bullet$, such that, $\forall \mathfrak{X}_1,\dots,\mathfrak{X}_n \in \mathrm{Der}_\varepsilon^\bullet(\mathbf{A})$ homogeneous,

Definition 5 ($\underline{\Omega}_\varepsilon^{n,k}(\mathbf{A},\mathbf{M})$)

$$\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_n) \in \mathbf{M}^{k+|\mathfrak{X}_1|+\dots+|\mathfrak{X}_n|},$$
$$\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_n z) = \omega(\mathfrak{X}_1,\dots,\mathfrak{X}_n)z,\ \forall z \in \mathcal{Z}_\varepsilon^\bullet(\mathbf{A})$$
$$\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_i,\mathfrak{X}_{i+1},\dots,\mathfrak{X}_n) = -\varepsilon(|\mathfrak{X}_i|,|\mathfrak{X}_{i+1}|)\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_{i+1},\mathfrak{X}_i,\dots,\mathfrak{X}_n)$$

We note $\underline{\Omega}_\varepsilon^{0,k}(\mathbf{A},\mathbf{M}) = \mathbf{M}^k$ and $\underline{\Omega}_\varepsilon^{\bullet,\bullet}(\mathbf{A}) = \underline{\Omega}_\varepsilon^{\bullet,\bullet}(\mathbf{A},\mathbf{A})$ if $\mathbf{M}^\bullet = \mathbf{A}^\bullet$

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- $\underline{\Omega}_{\varepsilon}^{\bullet,\bullet}(\mathbf{A})$ is a differential algebra for the product and the differential defined in the following slide
- $\underline{\Omega}_{\varepsilon}^{\bullet,\bullet}(\mathbf{A})$ is an $\widetilde{\varepsilon}$-graded algebra for the abelian group $\widetilde{\Gamma} = \mathbb{Z} \times \Gamma$ with $\widetilde{\varepsilon}((p,i),(q,j)) = (-1)^{pq}\varepsilon(i,j)$

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- d is an $\widetilde{\varepsilon}$-derivation of $\underline{\Omega}_\varepsilon^{\bullet,\bullet}(\mathbf{A})$ of degree $(1,0)$ satisfying $\mathrm{d}^2 = 0$

Differential calculus based on ϵ-derivations

- The product is : $(\omega\cdot\eta)(\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q})$
$$= \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{|\sigma|} f_1\ \omega(\mathfrak{X}_{\sigma(1)},\dots,\mathfrak{X}_{\sigma(p)})\cdot\eta(\mathfrak{X}_{\sigma(p+1)},\dots,\mathfrak{X}_{\sigma(p+q)}),$$
- The differential is
$$\mathrm{d}\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_{p+1}) = \sum_{m=1}^{p+1}(-1)^{m+1} f_2\ \mathfrak{X}_m\omega(\mathfrak{X}_1,\dots\overset{m}{\vee}\dots,\mathfrak{X}_{p+1})$$
$$+\sum_{1\leq m<n\leq p+1}(-1)^{m+n} f_3\ \omega([\mathfrak{X}_m,\mathfrak{X}_n]_\varepsilon,\dots\overset{m}{\vee}\dots\overset{n}{\vee}\dots,\mathfrak{X}_{p+1})$$
- The factors f_i are given by
$f_1 = \prod_{m<n,\sigma(m)>\sigma(n)} \varepsilon(|\mathfrak{X}_{\sigma(n)}|,|\mathfrak{X}_{\sigma(m)}|)\prod_{m\leq p}\varepsilon(|\eta|,|\mathfrak{X}_{\sigma(m)}|)$
$f_2 = \varepsilon(|\omega|,|\mathfrak{X}_m|)\prod_{a=1}^{m-1}\varepsilon(|\mathfrak{X}_a|,|\mathfrak{X}_m|)$
$f_3 = \varepsilon(|\mathfrak{X}_n|,|\mathfrak{X}_m|)\prod_{a=1}^{m-1}\varepsilon(|\mathfrak{X}_a|,|\mathfrak{X}_m|)\prod_{a=1}^{n-1}\varepsilon(|\mathfrak{X}_a|,|\mathfrak{X}_n|)$
$\omega\in\underline{\Omega}_\varepsilon^{p,|\omega|}(\mathbf{A})$, $\eta\in\underline{\Omega}_\varepsilon^{q,|\eta|}(\mathbf{A})$, and $\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q}\in\mathrm{Der}_\varepsilon^\bullet(\mathbf{A})$ homogeneous

Cartan operation

- An ε-Lie subalgebra $\mathfrak{g}^\bullet$ of $\mathrm{Der}^\bullet_\varepsilon(\mathbf{A})$ defines canonically a Cartan operation on $(\underline{\Omega}^{\bullet,\bullet}_\varepsilon(\mathbf{A}), \mathrm{d})$ by :
  The inner product with $\mathfrak{X} \in \mathrm{Der}^\bullet_\varepsilon(\mathbf{A})$ is the map
  $i_\mathfrak{X} : \underline{\Omega}^{n,k}_\varepsilon(\mathbf{A}) \to \underline{\Omega}^{n-1,k+|\mathfrak{X}|}_\varepsilon(\mathbf{A})$
  $$i_\mathfrak{X}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}) = \varepsilon(|\mathfrak{X}|, |\omega|)\omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_{n-1})$$
  and $i_\mathfrak{X}\underline{\Omega}^{0,\bullet}_\varepsilon(\mathbf{A}) = 0$. $i_\mathfrak{X}$ is an $\widetilde{\varepsilon}$-derivation of $\underline{\Omega}^{\bullet,\bullet}_\varepsilon(\mathbf{A})$ of degree $(-1, |\mathfrak{X}|)$.
  The associated Lie derivative
  $L_\mathfrak{X} = [i_\mathfrak{X}, \mathrm{d}] = i_\mathfrak{X}\mathrm{d} + \mathrm{d}i_\mathfrak{X} : \underline{\Omega}^{n,k}_\varepsilon(\mathbf{A}) \to \underline{\Omega}^{n,k+|\mathfrak{X}|}_\varepsilon(\mathbf{A})$, is an $\widetilde{\varepsilon}$-derivation of $\underline{\Omega}^{\bullet,\bullet}_\varepsilon(\mathbf{A})$ of degree $(0, |\mathfrak{X}|)$

Proposition 6

$$[i_\mathfrak{X}, i_\mathfrak{Y}] = i_\mathfrak{X} i_\mathfrak{Y} + \varepsilon(|\mathfrak{X}|, |\mathfrak{Y}|) i_\mathfrak{Y} i_\mathfrak{X} = 0$$
$$[L_\mathfrak{X}, i_\mathfrak{Y}] = L_\mathfrak{X} i_\mathfrak{Y} - \varepsilon(|\mathfrak{X}|, |\mathfrak{Y}|) i_\mathfrak{Y} L_\mathfrak{X} = i_{[\mathfrak{X},\mathfrak{Y}]_\varepsilon}$$
$$[L_\mathfrak{X}, \mathrm{d}] = L_\mathfrak{X}\mathrm{d} - \mathrm{d}L_\mathfrak{X} = 0$$
$$[L_\mathfrak{X}, L_\mathfrak{Y}] = L_\mathfrak{X} L_\mathfrak{Y} - \varepsilon(|\mathfrak{X}|, |\mathfrak{Y}|) L_\mathfrak{Y} L_\mathfrak{X} = L_{[\mathfrak{X},\mathfrak{Y}]_\varepsilon}$$

ϵ-connection and curvature

Definition 7 (ε-connection)

An homogeneous linear map of degree 0, $\nabla : \mathbf{M}^\bullet \to \underline{\Omega}_\varepsilon^{1,\bullet}(\mathbf{A},\mathbf{M})$ is called an ε-connection if $\forall a \in \mathbf{A}^\bullet$, $\forall m \in \mathbf{M}^\bullet$,

$$\nabla(ma) = \nabla(m)a + m\mathrm{d}a \tag{1}$$

- We can extend ∇ as a linear map $\nabla : \underline{\Omega}_\varepsilon^{p,\bullet}(\mathbf{A},\mathbf{M}) \to \underline{\Omega}_\varepsilon^{p+1,\bullet}(\mathbf{A},\mathbf{M})$

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Definition 8 (curvature)

We define the curvature $R = \nabla^2$, homogeneous linear map of degree 0, $\forall m \in \mathbf{M}^\bullet$, $\forall \mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}_\varepsilon^\bullet(\mathbf{A})$ homogeneous,

$$R(m)(\mathfrak{X}, \mathfrak{Y}) = \varepsilon(|\mathfrak{X}|, |\mathfrak{Y}|)\nabla(\nabla(m)(\mathfrak{Y}))(\mathfrak{X}) - \nabla(\nabla(m)(\mathfrak{X}))(\mathfrak{Y}) - \nabla(m)([\mathfrak{X}, \mathfrak{Y}]_\varepsilon)$$

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Definition 9 (gauge transformations)

The gauge transformations of $\mathbf{M}^\bullet$ are the automorphisms of degree 0 of $\mathbf{M}^\bullet$.

They form the gauge group $\mathrm{Aut}^0_{\mathbf{A}}(\mathbf{M}, \mathbf{M})$ and act on the space of its ε-connections : $\forall \Phi \in \mathrm{Aut}^0_{\mathbf{A}}(\mathbf{M}, \mathbf{M})$, $\nabla^\Phi = \Phi \circ \nabla \circ \Phi^{-1}$ is an ε-connections

Examples

Examples

- ε-graded matrix algebras (see in 0811.3567 [math-ph])
- Freely finitely generated abelian groups
- Quillen connection
- Clifford algebra
- Extension to Hom algebras (0811.0400 [math.RA])

A simple application

The Moyal algebra

- Deformed Moyal product defined on the space of Schwartz functions $\mathcal{S}$:

$$(f \star_\theta g)(x) = \frac{1}{\pi^4\theta^4}\int d^4y d^4z\, f(x+y)g(x+z)e^{-2iy\Theta^{-1}z}$$

$$\Theta = \begin{pmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta \\ 0 & 0 & \theta & 0 \end{pmatrix}$$

- The Moyal algebra is defined by $\mathcal{M}_\theta = \mathcal{L} \cap \mathcal{R}$ where

$$\mathcal{L} = \{T \in \mathcal{S}' / f \star_\theta T \in \mathcal{S}, \forall f \in \mathcal{S}\}$$

$$\mathcal{R} = \{T \in \mathcal{S}' / T \star_\theta f \in \mathcal{S}, \forall f \in \mathcal{S}\}$$

see e.g. *Algebras of distributions suitable for phase-space quantum mechanics* Gracia-Bondia, Varilly and ref therein

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- Deformed Moyal product defined on the space of Schwartz functions $\mathcal{S}$:

$$(f \star_\theta g)(x) = \frac{1}{\pi^4\theta^4} \int d^4y d^4z\, f(x+y)g(x+z)e^{-2iy\Theta^{-1}z}$$

$$\Theta = \begin{pmatrix} 0 & -\theta & & \\ \theta & 0 & & \\ & & 0 & -\theta \\ & & \theta & 0 \end{pmatrix}$$

- The Moyal algebra is defined by $\mathcal{M}_\theta = \mathcal{L} \cap \mathcal{R}$ where

$$\mathcal{L} = \{T \in \mathcal{S}' / f \star_\theta T \in \mathcal{S}, \forall f \in \mathcal{S}\}$$

$$\mathcal{R} = \{T \in \mathcal{S}' / T \star_\theta f \in \mathcal{S}, \forall f \in \mathcal{S}\}$$

  see e.g. *Algebras of distributions suitable for phase-space quantum mechanics* Gracia-Bondia, Varilly and ref therein

- We define

$$\gamma = 1 \qquad \xi_\mu = -\frac{1}{2}\widetilde{x}_\mu \qquad \eta_{\mu\nu} = \frac{1}{2}\widetilde{x}_\mu\widetilde{x}_\nu = 2\xi_\mu\xi_\nu$$

where $\widetilde{x}_\mu = 2\Theta^{-1}_{\mu\nu}x_\nu$

A $\mathbb{Z}_2$-graded algebra constructed from Moyal space

Definition 10

Let $\mathbf{A}^\bullet = \mathcal{M}_\theta \oplus \mathcal{M}_\theta$ be the $\mathbb{Z}_2$-graded complex vector space defined by the following product (whith $\alpha \in \mathbb{R}$) : $\forall \phi, \psi \in \mathbf{A}^\bullet$,

$$\phi \cdot \psi = (\phi_0, \phi_1) \cdot (\psi_0, \psi_1) = (\phi_0 \star \psi_0 + \alpha\ \phi_1 \star \psi_1, \phi_0 \star \psi_1 + \phi_1 \star \psi_0) \tag{2}$$

The commutation factor is defined by $\varepsilon(i,j) = (-1)^{ij}$ for $i, j \in \mathbb{Z}_2$

A $\mathbb{Z}_2$-graded algebra constructed from Moyal space

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- The bracket $[\phi, \psi]_\varepsilon = ([\phi_0, \psi_0]_\star + \alpha\{\phi_1, \psi_1\}_\star, [\phi_0, \psi_1]_\star + [\phi_1, \psi_0]_\star)$
- The ε-center of $\mathbf{A}^\bullet$ is $\mathcal{Z}^\bullet_\varepsilon(\mathbf{A}) = \mathbb{C}\mathbb{1} = \mathbb{C} \oplus 0$ where the unit $\mathbb{1}$ is $(1, 0)$
- Trace on $\mathbf{A}^\bullet$: $\text{Tr}(\phi) = \text{Tr}(\phi_0, \phi_1) = \int d^4x\ \phi_0(x)$
- ε-Lie subalgebra $\mathfrak{g}^\bullet$ of $\text{Der}^\bullet_\varepsilon(\mathbf{A})$:
  $$\mathfrak{g}^\bullet = \langle [(0, i\gamma), \cdot],\ [(i\xi_\mu, 0), \cdot],\ [(0, i\xi_\mu), \cdot],\ [(i\eta_{\mu\nu}, 0), \cdot] \rangle$$
  Natural generalisation of the extended algebra of derivation of Moyal (0804.3061 [hep-th])
- Γ-graded module $\mathbf{M}^\bullet = \mathbf{A}^\bullet$

Connection

Proposition 11

An a ε-connection ∇ on $\mathbf{A}^\bullet$ and a ε-derivation was associated to a gauge potential $A_{\mathfrak{X}}$ defined by $-iA_{\mathfrak{X}} = \nabla(\mathbb{1})(\mathfrak{X})$.
The ε-connection ∇ becomes : $\forall a \in \mathbf{A}^\bullet$, $\nabla_{\mathfrak{X}} a = \mathfrak{X}(a) - iA_{\mathfrak{X}} \cdot a$ where $\nabla_{\mathfrak{X}} a = \varepsilon(|\mathfrak{X}|, |a|)\nabla(a)(\mathfrak{X})$.
Defining $F_{\mathfrak{X},\mathfrak{Y}} \cdot a = i\varepsilon(|\mathfrak{X}| + |\mathfrak{Y}|, |a|)R(a)(\mathfrak{X},\mathfrak{Y})$, we obtain the curvature :

$$F_{\mathfrak{X},\mathfrak{Y}} = \mathfrak{X}(A_{\mathfrak{Y}}) - \varepsilon(|\mathfrak{X}|,|\mathfrak{Y}|)\mathfrak{Y}(A_{\mathfrak{X}}) - i[A_{\mathfrak{X}}, A_{\mathfrak{Y}}]_\varepsilon - A_{[\mathfrak{X},\mathfrak{Y}]_\varepsilon}. \tag{3}$$

- We note the gauge potentials :

$$\nabla(\mathbb{1})(\mathrm{ad}_{(0,i\gamma)}) = (0, -i\varphi), \qquad \nabla(\mathbb{1})(\mathrm{ad}_{(i\xi_\mu,0)}) = (-iA^0_\mu, 0),$$
$$\nabla(\mathbb{1})(\mathrm{ad}_{(0,i\xi_\mu)}) = (0, -iA^1_\mu), \qquad \nabla(\mathbb{1})(\mathrm{ad}_{(i\eta_{\mu\nu},0)}) = (-iG_{\mu\nu}, 0).$$

- The covariant coordinates are :

$$\Phi = \varphi - 1, \quad \mathcal{A}^0_\mu = A^0_\mu + \frac{1}{2}\widetilde{x}_\mu, \quad \mathcal{A}^1_\mu = A^1_\mu + \frac{1}{2}\widetilde{x}_\mu, \quad \mathcal{G}_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}\widetilde{x}_\mu\widetilde{x}_\nu$$

- $\mathcal{A}^1_\nu = \mathcal{A}^0_\nu$

$$
\begin{aligned}
F_{(0,i\gamma),(0,i\gamma)} &= (2i\alpha\varphi - 2i\alpha\varphi\star\varphi, 0)\\
F_{(i\xi_\mu,0),(0,i\gamma)} &= (0, \partial_\mu\varphi - i[A_\mu,\varphi]_\star)\\
F_{(0,i\xi_\mu),(0,i\gamma)} &= (-i\alpha(\widetilde{x}_\mu\varphi + \{A_\mu,\varphi\}_\star), 0)\\
F_{(i\eta_{\mu\nu},0),(0,i\gamma)} &= (0, -\frac{1}{4}\widetilde{x}_\mu\partial_\nu\varphi - \frac{1}{4}\widetilde{x}_\nu\partial_\mu\varphi - i[G_{\mu\nu},\varphi]_\star)\\
F_{(i\xi_\mu,0),(i\xi_\nu,0)} &= (\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu,A_\nu]_\star, 0)\\
F_{(i\xi_\mu,0),(0,i\xi_\nu)} &= (0, \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu,A_\nu]_\star - \Theta^{-1}_{\mu\nu}\varphi)\\
F_{(0,i\xi_\mu),(0,i\xi_\nu)} &= (-i\alpha\widetilde{x}_\mu A_\nu - i\alpha\widetilde{x}_\nu A_\mu - i\alpha\{A_\mu,A_\nu\}_\star - i\alpha G_{\mu\nu}, 0)\\
F_{(i\xi_\mu,0),(i\eta_{\nu\rho},0)} &= (\partial_\mu G_{\nu\rho} + \widetilde{x}_\nu\partial_\rho A_\mu + \widetilde{x}_\rho\partial_\nu A_\mu - i[A_\mu,G_{\nu\rho}]_\star\\
&\qquad + 2\Theta^{-1}_{\nu\mu}A_\rho + 2\Theta^{-1}_{\rho\mu}A_\nu, 0)\\
F_{(0,i\xi_\mu),(i\eta_{\nu\rho},0)} &= (0, \partial_\mu G_{\nu\rho} + \widetilde{x}_\nu\partial_\rho A_\mu + \widetilde{x}_\rho\partial_\nu A_\mu - i[A_\mu,G_{\nu\rho}]_\star\\
&\qquad + 2\Theta^{-1}_{\nu\mu}A_\rho + 2\Theta^{-1}_{\rho\mu}A_\nu)\\
F_{(i\eta_{\mu\nu},0),(i\eta_{\rho\sigma},0)} &= (-\widetilde{x}_\mu\partial_\nu G_{\rho\sigma} - \widetilde{x}_\nu\partial_\mu G_{\rho\sigma} + \widetilde{x}_\rho\partial_\sigma G_{\mu\nu} + \widetilde{x}_\sigma\partial_\rho G_{\mu\nu} - i[G_{\mu\nu},G_{\rho\sigma}]_\star\\
&\quad - 2\Theta^{-1}_{\mu\rho}G_{\nu\sigma} - 2\Theta^{-1}_{\nu\rho}G_{\mu\sigma} - 2\Theta^{-1}_{\mu\sigma}G_{\nu\rho} - 2\Theta^{-1}_{\nu\sigma}G_{\mu\rho}, 0)
\end{aligned}
$$

Gauge action

- The action $S = \mathrm{Tr}\left(|F_{\mathrm{ad}_a,\mathrm{ad}_b}|^2\right)$ involves
- $\mathcal{G}_{\mu\nu} = 0$, $\Phi = 0$

  $$\int d^D x\Big((1+2\alpha)F_{\mu\nu}\star F_{\mu\nu} + \alpha^2\{\mathcal{A}_\mu,\mathcal{A}_\nu\}_\star^2 + \frac{8}{\theta}(2(D+1)(1+\alpha)+\alpha^2)\mathcal{A}_\mu\star\mathcal{A}_\mu\Big)$$

  Gauge theory model of de Goursac, Wallet, Wulkenhaar (0703.075 [hep-th]) and Grosse, Wohlgenannt (0703.169 [hep-th])
- $[\mathcal{A}_\mu,\mathcal{A}_\nu]_\star \rightleftarrows \{\mathcal{A}_\mu,\mathcal{A}_\nu\}_\star$ natural interpretation stemming from $\mathbb{Z}_2$-grading

Gauge action

- $\mathcal{G}_{\mu\nu} = 0$, $\Phi \neq 0$

$$S(\varphi) = \int d^D x \Big(2\alpha |\partial_\mu \varphi - i[A_\mu, \varphi]_\star|^2 + 2\alpha^2 |\widetilde{x}_\mu \varphi + \{A_\mu, \varphi\}_\star|^2 - 4\alpha\sqrt{\theta}\varphi\Theta^{-1}_{\mu\nu}F_{\mu\nu} + \frac{2\alpha(D+2\alpha)}{\theta}\varphi^2 - \frac{8\alpha^2}{\sqrt{\theta}}\varphi\star\varphi\star\varphi + 4\alpha^2\varphi\star\varphi\star\varphi\star\varphi \Big)$$

  Grosse Wulkenhaar scalar model coupled to A_μ (0401.128 [hep-th])
  Harmonic term : natural interpretation stemming from F^2
  x_μ : canonical gauge invariant connection
  φ : composante of a gauge potential
- Slavnov term $-4\alpha\sqrt{\theta}\varphi\Theta^{-1}_{\mu\nu}F_{\mu\nu}$ (0304.141 [hep-th])